Letters

Perturbation Theory Generalized to Arbitrary (p,l) Modes in a Fabry–Perot Resonator

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Abstract—**The first-order perturbation calculation that was carried out to include the effects of the usually neglected $\partial^2\psi/\partial z^2$ term in the wave equation for pure radial modes ($l = 0$) is generalized to the case of arbitrary p,l modes. The result, although requiring more algebra, is similar to that of the $l = 0$ case; the correction term is a monotonically increasing function of both p and l, and reduces to the original expression as $l \to 0$.**

I. Introduction

As reported in [1], the resonance frequencies of high-order modes in a Fabry–Perot resonator depart from those predicted by the "large-aperture" approximation [2]. This paper extends the previously reported calculation, which applied only to radial modes, to arbitrary $\text{TEM}_{p,l}$ modes.

II. Theoretical Background

A scalar component u of the electric field satisfies the wave equation [2]

$$\nabla^2 u + k^2 u = 0. \tag{1}$$

When solutions of the form

$$u(r,\theta,z) = \psi(r,\theta,z)e^{-jkz} \tag{2}$$

are sought, (1) reduces to

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2} = 2jk\frac{\partial\psi}{\partial z} \tag{3}$$

and when the usual approximation, $\partial^2\psi/\partial z^2 \approx 0$, is made, the resulting approximate equation has the solution

$$\psi_p^l = \left(\sqrt{2}\frac{r}{w}\right)^l L_p^l\left(\frac{2r^2}{w^2}\right)\frac{w_0}{w}e^{-r^2/w^2}\cos l\theta \cdot \exp\left(j((2p+l+1)\tan^{-1}az - r^2/w^2az)\right) \tag{4}$$

where L_p^l is the generalized Laguerre polynominal and w_0 is the beam waist. R is the radius of curvature of the mirrors and d is the mirror spacing. The function w is defined as

$$w^2 = w_0^2[1 + a^2z^2]$$

where

$$a = \frac{\lambda}{\pi w_0^2}.$$

III. Results of the Perturbation Calculation

The procedure used to evaluate the resonance frequencies of high-order (p,l) modes is that described in [1] with the addition of θ dependence to the approximate solution. For the reader's convenience an explicit expression for the "matrix element" in the perturbation is given in the Appendix. With this the integration can be performed and the frequency shift evaluated to be

$$\Delta f = \frac{c}{\pi d}\tan^{-1}\left(\frac{a}{2k}\,\frac{6p^2 + 6p + 2 + 6pl + 3l + l^2}{4}\,\frac{ad}{2}\right) \tag{5}$$

where Δf is the first-order correction to the approximate resonant frequency. The expression reduces to that in [1] as $l \to 0$.

In the approximate theory the resonant frequency of a $\text{TEM}_{p,l}$ mode is given by

$$f = \frac{c}{2d}\left[q + \frac{2}{\pi}(2p + l + 1)\tan^{-1}\left(\frac{ad}{2}\right)\right] \tag{6}$$

i.e., modes with the same value of $2p + l$ are degenerate with respect to frequency. It is to be noted that the first-order correction removes this degeneracy.

Appendix

Direct differentiation of ψ_p^l with respect to z, including the z dependence in w, yields, after some algebraic manipulation,

$$\begin{aligned}\frac{1}{2jk}\frac{\partial^2\psi_p^l}{\partial z^2} = \frac{a^2}{2k}\Bigg\{ & -4j(p+l)(p+l-1)\frac{x^2}{X^2}\psi_{p-2}^l e^{4j\tan^{-1}x} \\ & + 2(p+l)\Bigg[2(2p+l+1)\frac{x}{X^2} \\ & + j\left\{(4p+2l+1)\frac{x^2}{X^2} - \frac{1}{X^2}\right\}\Bigg]\psi_{p-1}^l e^{2j\tan^{-1}x} \\ & + 2(p+l)\left[\frac{x^3}{X^2} - \frac{x}{X^2} - 2j\frac{x^2}{X^2}\right]\frac{2r^2}{w^2}\psi_{p-1}^l e^{2j\tan^{-1}x} \\ & + (2p+l+1)(2p+l+2)\left[\frac{-2x}{X^2} + j\left\{\frac{1}{X^2} - \frac{x^2}{X^2}\right\}\right]\psi_p^l \\ & + (2p+l+2)\left[\frac{3x}{X^2} - \frac{x^3}{X^2} + j\left\{\frac{3x^2}{X^2} - \frac{1}{X^2}\right\}\right]\frac{2r^2}{w^2}\psi_p^l \\ & + \left[\frac{x^3}{X^2} - \frac{x}{X^2} + j\left\{\frac{1}{4} - \frac{2x^2}{X^2}\right\}\right]\left(\frac{2r^2}{w^2}\right)^2\psi_p^l\Bigg\}\end{aligned}$$

where $x = aZ$ and $X = 1 + x^2$.

References

[1] C. W. Erickson, "High order modes in a spherical Fabry–Perot resonator," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 218–223, Feb. 1975.
[2] H. Kogelnik and T. Li, "Laser beams and resonators," *Appl. Opt.*, vol. 5, pp. 1550–1567, Oct. 1966.

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Correction to "Coupled Transmission Lines"

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In the above paper,[1] the sentence following equation (6) should read: where $(\bar{I})$ is a 90° rotation dyadic given by.

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[1] M. Kumar and B. N. Das, *IEEE Trans Microwave Theory Tech.*, vol. MTT-25, pp. 7–10, Jan. 1977.